

# Collective Origin of the Coexistence of Apparent RMT Noise and Factors in Large Sample Correlation Matrices

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Through simple analytical calculations and numerical simulations, we demonstrate the generic existence of a self-organized macroscopic state in any large multivariate system possessing non-vanishing average correlations between a finite fraction of all pairs of elements. The coexistence of an eigenvalue spectrum predicted by random matrix theory (RMT) and a few very large eigenvalues in large empirical correlation matrices is shown to result from a bottom-up collective effect of the underlying time series rather than a top-down impact of factors. Our results, in excellent agreement with previous results obtained on large financial correlation matrices, show that there is relevant information also in the bulk of the eigenvalue spectrum and rationalize the presence of market factors previously introduced in an *ad hoc* manner.

Since Wigner's seminal idea to apply random matrix theory (RMT) to interpret the complex spectrum of energy levels in nuclear physics [1], RMT has made enormous progress [2] with many applications in physical sciences and elsewhere such as in meteorology [3] and image processing [4]. A new application was proposed a few years ago to the problem of correlations between financial assets and to the portfolio optimization problem. It was shown that, among the eigenvalues and principal components of the empirical correlation matrix of the returns of hundreds of asset on the New York Stock Exchange (NYSE), apart from the few highest eigenvalues, the marginal distribution of the other eigenvalues and eigenvectors closely resembles the spectral distribution of a positive symmetric random matrix with maximum entropy, suggesting that the correlation matrix does not contain any specific information beyond these few largest eigenvalues and eigenvectors [5]. These results apparently invalidate the standard mean-variance portfolio optimization theory [6] consecrated by the financial industry [7] and seemingly support the rationale behind factor models such as the capital asset pricing model (CAPM) [8] and the arbitrage pricing theory (APT) [9], where the correlations between a large number of assets are represented through a small number of so-called market factors. Indeed, if the spectrum of eigenvalues of the empirical covariance or correlation matrices are predicted by RMT, it seems natural to conclude that there is no usable information in these matrices and that empirical covariance matrices should not be used for portfolio optimization. In contrast, if one detects deviations between the universal – and therefore non-informative – part of the spectral properties of empirically estimated covariance and correlation matrices and those of the relevant ensemble of random matrices, this may quantify the amount of real information that can be used in portfolio

optimization from the “noise” that should be discarded.

More generally, in many different scientific fields, one needs to determine the nature and amount of information contained in large covariance and correlation matrices. This occurs as soon as one attempts to estimate very large covariance and correlation matrices in multivariate dynamics of systems exhibiting non-Gaussian fluctuations with fat tails and/or long-range time correlations with intermittency. In such cases, the convergence of the estimators of the large covariance and correlation matrices is often too slow for all practical purposes. The problem becomes even more complex with time-varying variances and covariances as occurs in systems with heteroskedasticity [10] or with regime-switching [11]. A prominent example where such difficulties arise is the data-assimilation problem in engineering and in meteorology where forecasting is combined with observations iteratively through the Kalman filter, based on the estimation and forward prediction of large covariance matrices [12].

As we said in the context of financial time series, the rescuing strategy is to invoke the existence of a few dominant factors, such as an overall market factor and the factors related to firm size, firm industry and book-to-market equity, thought to embody most of the relevant dependence structure between the studied time series [13]. Indeed, there is no doubt that observed equity prices respond to a wide variety of unanticipated factors, but there is much weaker evidence that expected returns are higher for equities that are more sensitive to these factors, as required by Markowitz's mean-variance theory, by the CAPM and the APT [14]. This severe failure of the most fundamental finance theories could conceivably be attributable to an inappropriate proxy for the market portfolio, but nobody has been able to show that this is really the correct explanation. This remark constitutes

the crux of the problem: the factors invoked to model the cross-sectional dependence between assets are not known in general and are either postulated based on economic intuition in financial studies or obtained as black box results in the recent analyses using RMT [5].

Here, we show that the existence of factors results from a collective effect of the assets, similar to the emergence of a macroscopic self-organization of interacting microscopic constituents. For this, we unravel the general physical origin of the large eigenvalues of large covariance and correlation matrices and provide a complete understanding of the coexistence of features resembling properties of random matrices and of large “anomalous” eigenvalues. Through simple analytical calculations and numerical simulations, we demonstrate the generic existence of a self-organized macroscopic state in any large system possessing non-vanishing average correlations between a finite fraction of all pairs of elements.

Let us first consider a large system of size  $N$  with correlation matrix  $C$  in which every non-diagonal pairs of elements exhibits the same correlation coefficient  $C_{ij} = \rho$  for  $i \neq j$  and  $C_{ii} = 1$ . Its eigenvalues are

$$\lambda_1 = 1 + (N - 1)\rho \quad \text{and} \quad \lambda_{i \geq 2} = 1 - \rho \quad (1)$$

with multiplicity  $N - 1$  and with  $\rho \in (0, 1)$  in order for the correlation matrix to remain positive definite. Thus, in the thermodynamics limit  $N \rightarrow \infty$ , even for a weak positive correlation  $\rho \rightarrow 0$  (with  $\rho N \gg 1$ ), a very large eigenvalue appears, associated with the delocalized eigenvector  $v_1 = (1/\sqrt{N})(1, 1, \dots, 1)$ , which dominates completely the correlation structure of the system. This trivial example stresses that the key point for the emergence of a large eigenvalue is not the strength of the correlations, provided that they do not vanish, but the large size  $N$  of the system.

This result (1) still holds qualitatively when the correlation coefficients are all distinct. To see this, it is convenient to use a perturbation approach. We thus add a small random component to each correlation coefficient:

$$C_{ij} = \rho + \epsilon \cdot a_{ij} \quad \text{for } i \neq j, \quad (2)$$

where the coefficients  $a_{ij} = a_{ji}$  have zero mean, variance  $\sigma^2$  and are independently distributed (There are additional constraints on the support of the distribution of the  $a_{ij}$ 's in order for the matrix  $C_{ij}$  to remain positive definite with probability one). The determination of the eigenvalues and eigenfunctions of  $C_{ij}$  is performed using the perturbation theory developed in quantum mechanics [15] up to the second order in  $\epsilon$ . We find that the largest eigenvalue becomes

$$E[\lambda_1] = (N - 1)\rho + 1 + \frac{(N - 1)(N - 2)}{N^2} \cdot \frac{\epsilon^2 \sigma^2}{\rho} + \mathcal{O}(\epsilon^3) \quad (3)$$

while, at the same order, the corresponding eigenvector  $v_1$  remains unchanged. The degeneracy of the eigenvalue

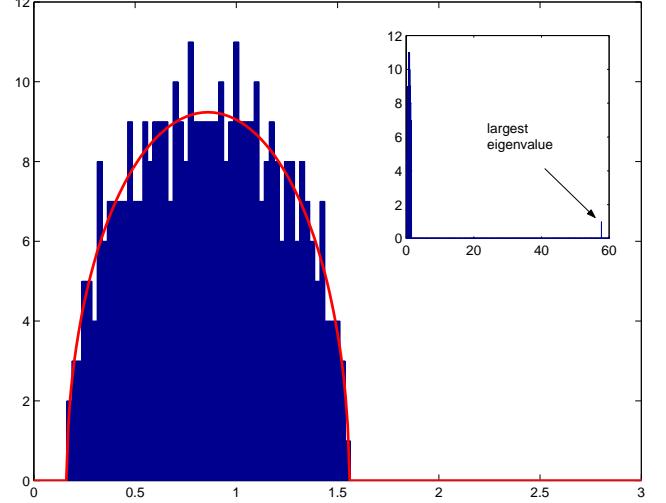


FIG. 1: Spectrum of eigenvalues of a random correlation matrix with average correlation coefficient  $\rho = 0.14$  and standard deviation of the correlation coefficients  $\sigma = 0.345/\sqrt{N}$ . The size  $N = 406$  of the matrix is the same as in previous studies [5] for the sake of comparison. The continuous curve is the theoretical translated semi-circle distribution of eigenvalues describing the bulk of the distribution which passes the Kolmogorov test. The center value  $\lambda = 1 - \rho$  ensures the conservation of the trace equal to  $N$ . There is no adjustable parameter. The inset represents the whole spectrum with the largest eigenvalue whose size is in agreement with the prediction  $\rho N = 56.8$ .

$\lambda = 1 - \rho$  is broken and leads to a complex set of smaller eigenvalues described below.

In fact, this result (3) can be generalized to the non-perturbative domain of any correlation matrix with independent random coefficients  $C_{ij}$ , provided that they have the same mean value  $\rho$  and variance  $\sigma^2$ . Indeed, it has been shown [16] that, in such a case, the expectations of the largest and second largest eigenvalues are

$$E[\lambda_1] = (N - 1) \cdot \rho + 1 + \sigma^2/\rho + o(1), \quad (4)$$

$$E[\lambda_2] \leq 2\sigma\sqrt{N} + \mathcal{O}(N^{1/3} \log N). \quad (5)$$

Moreover, the statistical fluctuations of these two largest eigenvalues are asymptotically (for large fluctuations  $t > \mathcal{O}(\sqrt{N})$ ) bounded by a Gaussian distribution according to the following large deviation theorem

$$\Pr\{|\lambda_{1,2} - E[\lambda_{1,2}]| \geq t\} \leq e^{-c_{1,2}t^2}, \quad (6)$$

for some positive constant  $c_{1,2}$  [17].

This result is very different from that obtained when the mean value  $\rho$  vanishes. In such a case, the distribution of eigenvalues of the random matrix  $C$  is given by the semi-circle law [2]. However, due to the presence of the ones on the main diagonal of the correlation matrix  $C$ , the center of the circle is not at the origin but at the

point  $\lambda = 1$ . Thus, the distribution of the eigenvalues of random correlation matrices with zero mean correlation coefficients is a semi-circle of radius  $2\sigma\sqrt{N}$  centered at  $\lambda = 1$ .

The result (4) is deeply related to the so-called “friendship theorem” in mathematical graph theory, which states that, in any finite graph such that any two vertices have exactly one common neighbor, there is one and only one vertex adjacent to all other vertices [18]. A more heuristic but equivalent statement is that, in a group of people such that any pair of persons have exactly one common friend, there is always one person (the “politician”) who is the friend of everybody. The connection is established by taking the non-diagonal entries  $C_{ij}$  ( $i \neq j$ ) equal to Bernouilli random variable with parameter  $\rho$ , that is,  $Pr[C_{ij} = 1] = \rho$  and  $Pr[C_{ij} = 0] = 1 - \rho$ . Then, the matrix  $C_{ij} - I$ , where  $I$  is the unit matrix, becomes nothing but the adjacency matrix of the random graph  $G(N, \rho)$  [17]. The proof of [18] of the “friendship theorem” indeed relies on the  $N$ -dependence of the largest eigenvalue and on the  $\sqrt{N}$ -dependence of the second largest eigenvalue of  $C_{ij}$  as given by (4) and (5).

Figure 1 shows the distribution of eigenvalues of a random correlation matrix. The inset shows the largest eigenvalue lying at the predicting size  $\rho N = 56.8$ , while the bulk of the eigenvalues are much smaller and are described by a modified semi-circle law centered on  $\lambda = 1 - \rho$ , in the limit of large  $N$ . The result on the largest eigenvalue emerging from the collective effect of the cross-correlation between all  $N(N-1)/2$  pairs provides a novel perspective to the observation [19] that the only reasonable explanation for the simultaneous crash of 23 stock markets worldwide in October 1987 is the impact of a world market factor: according to our demonstration, the simultaneous occurrence of significant correlations between the markets worldwide is bound to lead to the existence of an extremely large eigenvalue, the world market factor constructed by ... a linear combination of the 23 stock markets! What our result shows is that invoking factors to explain the cross-sectional structure of stock returns is cursed by the chicken-and-egg problem: factors exist because stocks are correlated; stocks are correlated because of common factors impacting them.

Figure 2 shows the eigenvalues distribution of the sample correlation matrix reconstructed by sampling  $N = 406$  time series of length  $T = 1309$  generated with a given correlation matrix  $C$  with theoretical spectrum shown in figure 1. The largest eigenvalue is again very close to the prediction  $\rho N = 56.8$  while the bulk of the distribution departs very strongly from the semi-circle law and is not far from the Wishart prediction, as expected from the definition of the Wishart ensemble as the ensemble of sample covariance matrices of Gaussian distributed time series with unit variance and zero mean. A Kolmogorov test shows however that the bulk of the spectrum is not in the Wishart class, in contradiction with previous claims

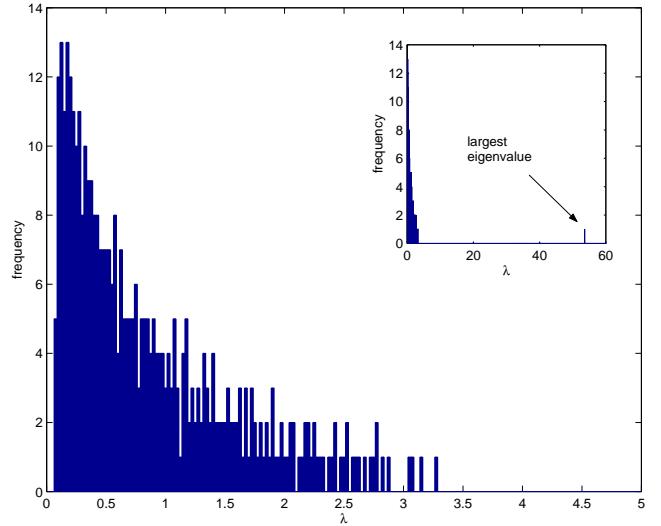


FIG. 2: Spectrum estimated from the sample correlation matrix obtained from  $N = 406$  time series of length  $T = 1309$  (the same length as in [5]) with the same theoretical correlation matrix as that presented in figure 1.

lacking formal statistical tests [5]. This result holds for different simulations of the sample correlation matrix and different realizations of the theoretical correlation matrix with the same parameters ( $\rho, \sigma$ ). The statistically significant departure from the Wishart prediction implies that there is actually some information in the bulk of the spectrum of eigenvalues, which is intimately coupled with the existence of the largest eigenvalue. We have also checked that these results remain robust for non-Gaussian distribution of returns as long as the second moments exist. Indeed, correlated time series with multivariate Gaussian or Student distributions with three degrees of freedom (which provide more acceptable proxies for financial time series [20]) give no discernible differences in the spectrum of eigenvalues. This is surprising as the estimator of a correlation coefficient is asymptotically Gaussian for time series with finite fourth moment and Lévy stable otherwise [21].

Up to now, we have focused on the collective mechanism at the origin of the very large eigenvalue of order  $N$ . Empirically [5], a few other eigenvalues have an amplitude of the order of 5 – 10 that deviate significantly from the bulk of the distribution. These eigenvalues cannot be obtained by a matrix of the form (2) with identically independently distributed coefficients  $a_{ij}$ 's. Our analysis provides a very simple constructive mechanism for them. The solution consists in considering, as a first approximation, the block diagonal matrix  $C'$  with diagonal elements made of the matrices  $A_1, \dots, A_p$  of sizes  $N_1, \dots, N_p$  with  $\sum N_i = N$ , constructed according to (2) such that each matrix  $A_i$  has the average correlation coefficient  $\rho_i$ . When the coefficients of the matrix  $C'$

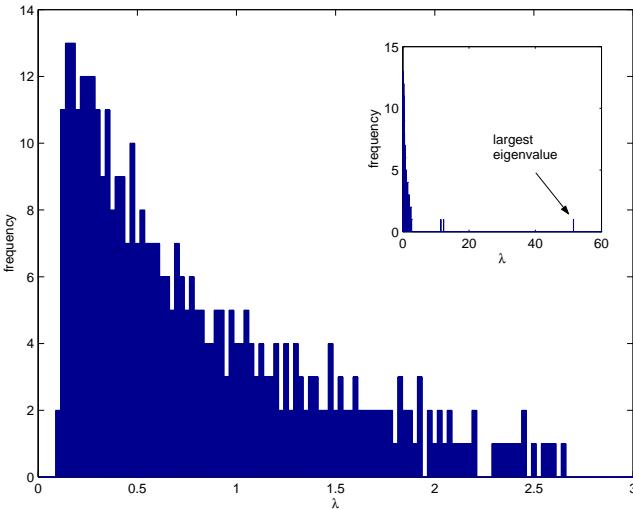


FIG. 3: Spectrum of eigenvalues estimated from the sample correlation matrix of  $N = 406$  time series of length  $T = 1309$ . The times series have been constructed from a multivariate Gaussian distribution with a correlation matrix made of three block-diagonal matrices of sizes respectively equal to 130, 140 and 136 and mean correlation coefficients equal to 0.18 for all of them. The off-diagonal elements are all equal to 0.1. The same results hold if the off-diagonal elements are random.

outside the matrices  $A_i$  are zero, the spectrum of  $C'$  is given by the union of all the spectra of the  $A_i$ 's, which are each dominated by a large eigenvalue  $\lambda_{1,i} \simeq \rho_i \cdot N_i$ . The spectrum of  $C'$  then exhibits  $p$  large eigenvalues.

Each block  $A_i$  can be interpreted as a sector of the economy, including all the companies belonging to a same industrial branch and the eigenvector associated with each largest eigenvalue represents the main factor driving this sector of activity [22]. For similar sector sizes  $N_i$  and average correlation coefficients  $\rho_i$ , the largest eigenvalues are of the same order of magnitude. In order to recover a very large unique eigenvalue, we reintroduce some coupling constants outside the block diagonal matrices. A well-known result of perturbation theory in quantum mechanics states that such coupling leads to a repulsion between the eigenstates, which can be observed in figure 3 where  $C'$  has been constructed with three block matrices  $A_1$ ,  $A_2$  and  $A_3$  and non-zero off-diagonal coupling described in the figure caption. These values allow us to quantitatively replicate the empirical finding of Laloux *et al.* in [5], where the three first eigenvalues are approximately  $\lambda_1 \simeq 57$ ,  $\lambda_2 \simeq 10$  and  $\lambda_3 \simeq 8$ . The bulk of the spectrum (which excludes the three largest eigenvalues) is similar to the Wishart distribution but again statistically different from it as tested with a Kolmogorov test. There is thus significant deviation from the predictions of RMT not only for the largest eigenvalues but also in the bulk.

As a final remark, expressions (3,4) and our numerical tests for a large variety of correlation matrices show that

the delocalized eigenvector  $v_1 = (1/\sqrt{N})(1, 1, \dots, 1)$ , associated with the largest eigenvalue is extremely robust and remains (on average) the same for any large system. Thus, even for time-varying correlation matrices – as in finance with important heteroskedastic effects – the composition of the main factor remains almost the same. This can be seen as a generalized limit theorem reflecting the bottom-up organization of broadly correlated time series.

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